Estimation of the effects of counter-measures on different types of accidents in the presence of regression effects

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ESTIMATION OF THE EFFECTS OF COUNTERMEASURES ON DIFFERENT TYPES OF ACCIDENTS IN THE PRESENCE OF REGRESSION EFFECTS

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Abstract—In an earlier paper [Danielsson, 1986], we studied the problem of estimating the safety effect of a countermeasure on the expected number of accidents at road junctions when high-accident sites are selected for the study. Often, however, the countermeasure leads to varying effectiveness on different types of accidents. This paper is a generalization in that we estimate effects of countermeasures for each type of accident. A major result, for which empirical support is provided, is that the expected regression effect is the same for all types of accidents.

1. INTRODUCTION

In an earlier paper [Danielsson, 1986], we described the problem of the regression effect which can occur in before-and-after studies [for detailed descriptions see Brüde and Larsson, 1982; Hauer, 1980a, 1980b; Hauer, 1986; and Hauer and Persaud, 1983]. In the paper, we proposed a method of purely estimating the effects of countermeasures despite the presence of regression effects.

Unfortunately, there are in practice further complications with the regression effects. Assume, for example, that we wish to study the number of accidents at road junctions to determine whether a particular countermeasure has any (positive) effect. Naturally, the effectiveness of the countermeasure most often varies between different types of accident. The *total* effect (on all types of accidents) can then be interpreted as the average effect. One possible way to estimate this effect is obtained if junctions are selected at random and an estimate then made of the total (constant) effect. In many cases, however, only those junctions which are especially prone to accidents are selected, so that it is hardly possible to expect that the accident composition at the selected junctions is “normal.” An estimate of the total effect from such a sample may therefore be extremely misleading.

One way of avoiding this complication is to estimate the effect for each type of accident. If this can be done, it is then possible to estimate the total effect for any accident composition at the junctions.

2. FORMULATION OF THE PROBLEM

Consider a junction which under normal conditions has mean values $m_1, m_2, \dots, m_p$ per time period for p mutually exclusive types of accident. Examples of such types are (for cars) single-vehicle accidents, rear-end collisions, overtaking accidents, etc. When introducing a countermeasure, we assume that the expected numbers of accidents are changed to

$$\alpha_1 m_1, \alpha_2 m_2, \dots, \alpha_p m_p.$$

The total expected number of accidents

$$m = m_1 + m_2 + \dots + m_p$$

is then changed to αm , where

$$\alpha = \frac{\alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_p m_p}{m}.$$

We call $(1 - \alpha)$ the total safety effect of the countermeasure, while $(1 - \alpha_1), \dots, (1 - \alpha_p)$ are the effects on the different types of accidents.

If a junction is selected at random from a population of similar junctions, a correct estimate of α can be obtained by studying the effect of the countermeasure on the total number of accidents. The estimate of $(1 - \alpha)$ can then be regarded as the estimated effect of the countermeasure for an average junction. This is obviously not true if the junction is not selected at random. In such a case, it would be necessary to try to estimate the individual effects $1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_p$, and then to use these to estimate the total effect $1 - \alpha$ for a given junction.

Consider a period before the implementation of a countermeasure and let

X_j = the number of accidents of type j occurring at the junction studied,
 $j = 1, 2, \dots, p$.

We make the assumption that different X_j are independent and Poisson distributed with parameters $m_1, m_2, \dots, m_p$ respectively. We also define the total number of accidents

$$X = \sum_{j=1}^p X_j,$$

which is, of course, Poisson distributed with the parameter

$$m = \sum_{j=1}^p m_j.$$

We further assume that the junction has been selected for study because the total number of accidents is large, i.e. that

$$X \geq k.$$

The conditional distribution for the total number of accidents at a *selected* junction is thus what can be termed truncated Poisson [see Danielsson, 1986; Hauer, 1980b; and Hauer and Persaud, 1983] according to

$$p'_r(m) = P(X = r | X \geq k) = \frac{p_r(m)}{q_k(m)}; \quad r = k, k + 1, \dots \quad (1)$$

where

$$p_r(m) = e^{-m} \frac{m^r}{r!}; \quad q_k(m) = \sum_{r=k}^{\infty} p_r(m). \quad (2)$$

Now we are interested not only in the (conditional) distribution for X but mainly in the distribution for $(X_1, X_2, \dots, X_p)$ given that $X \geq k$.

The simultaneous distribution

$$p'(r_1, r_2, \dots, r_p) = P(X_1 = r_1, X_2 = r_2, \dots, X_p = r_p | X \geq k) \quad (3)$$

becomes

$$p'(r_1, \dots, r_p) = \frac{p_{r_1}(m_1) \cdot p_{r_2}(m_2) \cdot \dots \cdot p_{r_p}(m_p)}{q_k(m)}, \quad \begin{array}{l} \text{all } r_j = 0, 1, 2, \dots, \\ r_1 + r_2 + \dots + r_p \geq k. \end{array} \quad (4)$$

3. ESTIMATION OF EXPECTED REGRESSION EFFECTS AND COUNTERMEASURE EFFECTS FOR DIFFERENT TYPES OF ACCIDENT

Before studying the estimation procedures in detail, it can be instructive to consider a practical example.

3.1 A practical example illuminating the regression effects

In the example in Table 1, we consider rural junctions in Sweden which have been unaltered during the period 1977–1982. We study the total number of accidents but also the accidents involving personal injury. The years 1977–1979 are regarded as the before period, and the selection criterion ($X \geq k$) is based on the total number of accidents. The years 1980–1982 are regarded as the after period, and by comparing the before and after accidents we have estimates of the regression effects, since no countermeasure has been implemented at the junctions during 1977–1982.

The estimated regression effects in Table 1 are fairly high for different selection criteria. We can also notice that the total accident number is lower in the after period than in the before period. This is probably due to a general trend in the accident development. Therefore, the true regression effects are lower than those calculated in the table, but we have not made any corrections.

The results in Table 1 can be formulated in the following hypothesis: the regression effects are the same for all types of accidents as for the total number of accidents. Since an example is not a proof, this hypothesis is proved in the next section.

3.2 Expected regression effects for different types of accidents

In Danielsson [1986], we showed that the expected regression effect for the total number of accidents is

$$R = \frac{\mu_k(m) - m}{\mu_k(m)} = \frac{p_{k-1}(m)}{q_{k-1}(m)},$$

where

$$\mu_k(m) = E[X|X \geq k]. \quad (5)$$

Table 1. Estimated regression effects for the total number of accidents and accidents involving personal injury at rural junctions in Sweden

Junctions During 1977–1979 Which Had the Total Number of Accidents		Number of Accidents 1977–1979		Number of Accidents 1980–1982		Estimated Regression Effects for the Number of Accidents	
		Total (b)	Involving Personal Injury (b')	Total (a)	Involving Personal Injury (a')	Total (b – a)/b	Involving Personal Injury (b' – a')/b'
≥ 5	61	358	137	195	76	0.46	0.45
≥ 4	113	566	215	318	123	0.44	0.43
≥ 3	232	923	356	504	194	0.45	0.46
≥ 2	518	1495	582	816	314	0.45	0.46
≥ 1	1193	2170	841	1313	507	0.39	0.40
≥ 0	2390	2170	841	1887	715	0.13	0.15

We now want to determine the regression effect for accident type j , i.e.

$$R_j = \frac{\mu_k(m_j) - m_j}{\mu_k(m_j)},$$

where

$$\mu_k(m_j) = E[X_j | X \geq k]. \quad (6)$$

Conditional to $X = r$ it holds that X_j is binomial distributed with expected value $r \cdot m_j/m$. Therefore,

$$\begin{aligned} \mu_k(m_j) &= E[X_j | X \geq k] = \sum_{r=k}^{\infty} p'_r(m) E[X_j | X \geq k, X = r] \\ &= \sum_{r=k}^{\infty} p'_r(m) E[X_j | X = r] = \sum_{r=k}^{\infty} p'_r(m) \cdot r \frac{m_j}{m} \\ &= \frac{m_j}{m} E[X | X \geq k] = \frac{m_j}{m} \cdot m \frac{q_{k-1}(m)}{q_k(m)} = m_j \frac{q_{k-1}(m)}{q_k(m)}. \end{aligned} \quad (7)$$

By substituting (7) into (6) we get

$$R_j = 1 - \frac{q_k(m)}{q_{k-1}(m)} = \frac{p_{k-1}(m)}{q_{k-1}(m)} = R. \quad (8)$$

This means that the expected regression effect is the same for all types of accidents. The result may appear slightly surprising, but with a little afterthought it is natural and even intuitive. The “extra” accidents generated by the condition $X \geq k$ obviously have to be distributed proportionally to $m_1, m_2, \dots, m_p$ for the different types of accidents.

3.3 Estimation of the expected accident numbers for a single junction

In order to estimate the different m_j for a single junction, we construct the likelihood function for $x_1, x_2, \dots, x_p$ (which are the observed number of accidents of different types)

$$\mathcal{L} = p'(x_1, \dots, x_p)$$

We then seek the m_j which maximize [see (4)]

$$\begin{aligned} l &= \ln(\mathcal{L}) = \sum_{j=1}^p \ln(p_{x_j}(m_j)) - \ln(q_k(m)) \\ &= \sum_{j=1}^p x_j \ln m_j - \sum_{j=1}^p \ln(x_j!) - \ln\left(\sum_{r=k}^{\infty} \frac{m^r}{r!}\right). \end{aligned} \quad (9)$$

By differentiating (9) we obtain

$$\frac{\partial l}{\partial m_j} = \frac{x_j}{m_j} - \frac{\sum_{r=k}^{\infty} \frac{m^{r-1}}{(r-1)!}}{\sum_{r=k}^{\infty} \frac{m^r}{r!}} = \frac{x_j}{m_j} - \frac{q_{k-1}(m)}{q_k(m)}. \quad (10)$$

If the derivatives in (10) are set equal to 0 we obtain

$$x_j = m_j \frac{q_{k-1}(m)}{q_k(m)}; \quad j = 1, 2, \dots, p. \quad (11)$$

The right-hand term in (11) is, of course, the earlier calculated $E[X_j|X \geq k]$, i.e. the maximum likelihood (ML) estimates above are identical to the moment estimates.

We have shown earlier [Danielsson, 1986] that the ML estimate for m can be calculated from the equation

$$x = m \cdot \frac{q_{k-1}(m)}{q_k(m)}, \quad (12)$$

and this result is, of course, also obtained if the equations in (11) are summed.

From (11) and (12) we can state that

$$x_j = m_j \frac{q_{k-1}(m)}{q_k(m)} = m_j \cdot \frac{x}{m}, \quad (13)$$

i.e. if m_j^* , $j = 1, 2, \dots, p$, and m^* are the ML estimates, we obtain the following simple relations

$$m_j^* = m^* \cdot \frac{x_j}{x}. \quad (14)$$

To obtain the ML estimate of all m_j , we therefore need only solve *one* equation numerically [the ML equation for m in (12)]. The simple relations in (14) then give the required estimates.

Since according to (8) the regression effects are the same for all types of accidents, it is sufficient to estimate R according to

$$R^* = \frac{x - m^*}{x}. \quad (15)$$

Finally, we can also determine an intuitive estimate of m_j by analogy with Hauer's estimate of m [Hauer, 1980a, 1980b]. Hauer has proposed the following estimate of m

$$\hat{m} = \begin{cases} 0 & \text{if } x = k \\ x & \text{if } x > k. \end{cases} \quad (16)$$

In agreement with (16) we estimate m_j with

$$\hat{m}_j = \begin{cases} 0 & \text{if } x = k \\ x_j & \text{if } x > k. \end{cases} \quad (17)$$

The estimator $\hat{m}_j$ is unbiased since [see (7)]

$$\begin{aligned} E[\hat{m}_j] &= \mu_k(m_j) - p'_k(m)k \cdot \frac{m_j}{m} = m_j \left(\frac{q_{k-1}(m)}{q_k(m)} - \frac{k}{m} \frac{p_k(m)}{q_k(m)} \right) \\ &= m_j \left(\frac{q_{k-1}(m) - p_{k-1}(m)}{q_k(m)} \right) = m_j, \end{aligned}$$

where

$$kp_k(m) = mp_{k-1}(m). \quad (18)$$

3.4 Simultaneous estimates of regression and countermeasure effects

Assume that at n junctions we wish to introduce a countermeasure which for different types of accidents has the constant effects $1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_p$, respectively.

Regardless of the nature of the site, we assume that the effects are fixed. Of course, in many practical situations this is unrealistic because remedial measures interact with their environment. We apply the following notation for junction no. i :

X_{ji} = the number of accidents of type j before the countermeasure is implemented;
 Y_{ji} = the corresponding number of accidents after the countermeasure is implemented (during an equal duration of time);

$X_{\cdot i} = \sum_{j=1}^p X_{ji}$ = the total number of accidents in the before period;

$Y_{\cdot i} = \sum_{j=1}^p Y_{ji}$ = the total number of accidents in the after period.

In Section 2, we studied the distributions for variables of the type X_{ji} ($X_{\cdot i}$ is assumed to be truncated in k_i). We assume that Y_{ji} is Poisson distributed with expected values $\alpha_j \cdot m_{ji}$ where

m_{ji} = the expected number of accidents of type j before implementation at the junction studied.

Furthermore, it holds (with assumed independence between the Y_{ji}) that $Y_{\cdot i}$ is Poisson distributed with expected values $\alpha_{\cdot i} \cdot m_{\cdot i}$ where

$m_{\cdot i} = \sum_{j=1}^p m_{ji}$ = the expected total number of accidents before implementation at the junction,

$\alpha_{\cdot i} = \sum_{j=1}^p \frac{m_{ji}}{m_{\cdot i}} \alpha_j$ = the total effect of the countermeasure at junction no. i .

Now, the problem is to estimate the countermeasure parameters $\alpha_1, \dots, \alpha_p$ and $\alpha_{\cdot i}$ by using directly all the data x_{ji} and y_{ji} from all junctions. In the appendix the following maximum likelihood estimates (ML estimates) of α_j and m_{ji} are derived

$$\alpha_j = \frac{\sum_{i=1}^n y_{ji}}{\sum_{i=1}^n m_{ji}}; \quad j = 1, 2, \dots, p \quad (19)$$

$$m_{ji} \cdot \frac{q_{k_i-1}(m_{\cdot i})}{q_{k_i}(m_{\cdot i})} + \alpha_j m_{ji} = x_{ji} + y_{ji}; \quad \begin{matrix} j = 1, 2, \dots, p \\ i = 1, 2, \dots, n. \end{matrix} \quad (20)$$

This complicated equation system can be solved by a rather simple numerical method, which is outlined in the appendix.

We observe that the estimator of α_j in (19) is of the same type of quotient as in the case where a common effect for all accidents is estimated [see Danielsson, 1986]. The argument in Danielsson [1986] for judging the uncertainty of α_j^* will be fully equivalent also here. We will therefore not concern ourselves with these details.

Finally, we note that it is also possible to estimate α_j by using the intuitive estimates of m_{ji} . The reasonable estimate of α_j is of course

$$\hat{\alpha}_j = \frac{\sum_{i=1}^n y_{ji}}{\sum_{i=1}^n \hat{m}_{ji}}, \quad (21)$$

where

$$\hat{m}_{ji} = \begin{cases} 0 & \text{if } x_{.i} = k_i \\ x_{ji} & \text{if } x_{.i} > k_i. \end{cases}$$

5. COMMENTS ABOUT THE PRACTICAL APPLICABILITY OF THE ESTIMATION METHODS

In the preceding section we have derived the ML estimators of the different α_j . The estimators are rather complicated to handle and there is a need for simpler and quicker estimation methods even though they are less efficient. We begin the discussion with the ML estimators and then successively introduce some simplifications.

(i) *ML estimators.* The estimator of α_j derived from (19) and (20) is rather complicated, even if the multidimensional problem can be reduced to a sequence of one-dimensional problems as described in the appendix.

The form of the ML estimator (19) is simple provided the estimators of m_{ji} are determined. Thus, simplified estimators can be obtained by replacing the denominator in (19) with estimators of m_{ji} other than the ML estimators.

(ii) *ML-like estimators.* In (21) we have proposed a simple estimator of α_j which does not require any complex calculation. The estimator of α_j can be formulated

$\hat{\alpha}_j$ = the total number of type j accidents in the after period divided by the corresponding total number in the before period for the sites which fulfill the selection criteria plus one type j accident.

Another estimator can be derived from (14). Using only the before data for estimating m_{ji} , the ML estimator is

$$m_{ji}^* = m_{.i}^* \frac{x_{ji}}{x_{.i}}, \quad (22)$$

where $m_{.i}^*$ easily can be calculated from [see (12)]

$$x_{.i} = m_{.i} \frac{q_{k_i-1}(m_{.i})}{q_{k_i}(m_{.i})}. \quad (23)$$

Then, the estimator of α_j is

$$\alpha_j^* = \frac{\sum_{i=1}^n y_{ji}}{\sum_{i=1}^n m_{ji}^*}. \quad (24)$$

(iii) *Estimators by explicit use of the regression effect.* If the regression effect R_i in (5) for the total number of accidents at each site is known, then α_j can be estimated from

$$\tilde{\alpha}_j = \frac{\sum_{i=1}^n y_{ji}}{\sum_{i=1}^n (1 - R_i)x_{ji}}. \quad (25)$$

The result follows from the definitions (5) and (6) and from the fact that all accident types have the same regression effect [see (8)].

In practice, the different R_i very seldom are known. However, in some cases it is possible to assess the regression effect and the estimate in (25) can be determined with this estimate of R_i . One way to estimate R_i is to use the "Strike method" proposed by Brüde and Larsson [1982]. Formula (25) is especially simple when all sites have the same regression effect R . Then we have

$$\tilde{\alpha}_j = \frac{\sum_{i=1}^n y_{ji}}{(1 - R) \sum_{i=1}^n x_{ji}}. \quad (26)$$

(iv) *An illustrative example.* From the material in Table 1, 10 junctions are selected, all with total accident numbers of at least 5 in the before period. The selected junctions are presented in Table 2a.

Let $1 - \alpha_1$, $1 - \alpha_2$, and $1 - \alpha$ be the countermeasure effects for accidents involving personal injury, accidents not involving personal injury, and all the accidents respectively. It is interesting to compare the estimates $\hat{\alpha}_j$, α_j^* and $\tilde{\alpha}_j$ proposed in (21), (24), and (26). It is worth noting that there is no real countermeasure effect, but instead we seek to estimate a general downward effect in the whole population of about 15%.

It is very easy to compute the estimates proposed in (21). Consider e.g. the accidents involving personal injury. Then

$$\hat{\alpha}_1 = \frac{14}{24 - 5} = 0.74.$$

To compute α_1^* in (24) it is necessary to solve the equation

$$10 = m_{.1}^* \frac{q_4(m_{.1}^*)}{q_5(m_{.1}^*)}$$

for junction No. 1, and then the corresponding equations for the other junctions. The definition of q_k can be found in (2). The solution of the equation above is

$$m_{.1}^* = 9.8,$$

and then

$$m_{11}^* = 9.8 \cdot \frac{4}{10} = 3.92.$$

Table 2a. Number of accidents for selected junctions divided with respect to personal injury

Junction No.	Number of Accidents 1977-1979			Number of Accidents 1980-1982		
	Total	Involving Personal Injury	Not Involving Personal Injury	Total	Involving Personal Injury	Not Involving Personal Injury
1	10	4	6	3	1	2
2	8	5	3	8	5	3
3	7	3	4	3	2	1
4	7	2	5	3	0	3
5	6	2	4	2	1	1
6	6	3	3	5	2	3
7	5	1	4	3	1	2
8	5	1	4	4	1	3
9	5	1	4	3	1	2
10	5	2	3	2	0	2
All junctions	64	24	40	36	14	22

Table 2b. Countermeasure effects estimated with different methods

	Estimated Countermeasure Effect for the Number of Accidents		
	Total	Involving Personal Injury	Not Involving Personal Injury
No correction method (α')	0.44	0.42	0.45
Hauer type of correction ($\hat{\alpha}$)	0.18	0.26	0.12
ML-type of correction (α^*)	0.02	0.12	-0.06
"Known" regression effect ($\tilde{\alpha}$)	0.13	0.10	0.15

For the observations 8, 7, 6, and 5 the corresponding m^* -values are 7.4, 6.0, 3.8, and 0. Then it is easy to compute all m_{li}^* and we have $\Sigma m_{li}^* = 16.0$. Therefore

$$\alpha_1^* = \frac{14}{16.0} = 0.88.$$

From Table 1, we can conclude that the regression effect is about 35% (since there is general downward secular trend of about 15%). Therefore we have from (26) (assuming the same regression effect at all sites)

$$\tilde{\alpha}_1 = \frac{14}{0.65 \cdot 24} = 0.90.$$

Without any compensation of the regression effect, the estimate of α_1 is

$$\alpha_1' = \frac{14}{24} = 0.58.$$

The results of all computations are presented in Table 2b. It is obvious that the countermeasure effects (in this case, the secular changes) are seriously overestimated if no correction method is applied. The various correction methods give similar results, but the estimates are rather unstable. An exception is the estimate $\tilde{\alpha}$, which, since it utilizes the "known" regression effect, is quite reasonable.

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APPENDIX

Maximum likelihood estimates of α_j and m_{ji} in Section 3.4

We wish to estimate the countermeasure parameters $\alpha_1, \dots, \alpha_p$ and α_i by directly using data from all n junctions. The total likelihood for the observed values x_{ji} and y_{ji} ($j = 1, 2, \dots, p, i = 1, 2, \dots, n$) will

be [see (2) and (4)]

$$\begin{aligned}
 \mathcal{L} &= \prod_{i=1}^n p'(x_{1i}, x_{2i}, \dots, x_{pi}) \cdot p_{y_{1i}}(\alpha_1 m_{1i}) \cdot \dots \cdot p_{y_{pi}}(\alpha_p m_{pi}) \\
 l = \ln(\mathcal{L}) &= \sum_{i=1}^n \left\{ \sum_{j=1}^p \ln(p_{x_{ji}}(m_{ji})) - \ln(q_{k_i}(m_{\cdot i})) + \sum_{j=1}^p \ln(p_{y_{ji}}(\alpha_j m_{ji})) \right\} \\
 &= \sum_{i=1}^n \left\{ \sum_{j=1}^p x_{ji} \ln m_{ji} - \sum_{j=1}^p \ln(x_{ji}!) - \ln\left(\sum_{r=k_i}^{\infty} \frac{(m_{\cdot i})^r}{r!}\right) \right. \\
 &\quad \left. - \sum_{j=1}^p \alpha_j m_{ji} + \sum_{j=1}^p y_{ji} \ln(\alpha_j \cdot m_{ji}) - \sum_{j=1}^p \ln(y_{ji}!) \right\}.
 \end{aligned} \tag{A1}$$

Differentiating (A1) gives

$$\begin{cases} \frac{\partial l}{\partial \alpha_j} = - \sum_{i=1}^n m_{ji} + \sum_{i=1}^n \frac{y_{ji}}{\alpha_j}; & j = 1, 2, \dots, p \\ \frac{\partial l}{\partial m_{ji}} = \frac{x_{ji}}{m_{ji}} - \frac{q_{k_i-1}(m_{\cdot i})}{q_{k_i}(m_{\cdot i})} - \alpha_j + \frac{y_{ji}}{m_{ji}}; & j = 1, 2, \dots, p, \\ & i = 1, 2, \dots, n. \end{cases} \tag{A2}$$

If the derivatives in (A2) are set equal to 0 we obtain

$$\alpha_j = \frac{\sum_{i=1}^n y_{ji}}{\sum_{i=1}^n m_{ji}}; \quad j = 1, 2, \dots, p, \tag{A3}$$

$$m_{ji} \cdot \frac{q_{k_i-1}(m_{\cdot i})}{q_{k_i}(m_{\cdot i})} + \alpha_j m_{ji} = x_{ji} + y_{ji}; \quad \begin{matrix} j = 1, 2, \dots, p \\ i = 1, 2, \dots, n. \end{matrix} \tag{A4}$$

Summing (A4) with regard to j , we obtain

$$m_{\cdot i} \cdot \frac{q_{k_i-1}(m_{\cdot i})}{q_{k_i}(m_{\cdot i})} + \sum_{j=1}^p \alpha_j m_{ji} = x_{\cdot i} + y_{\cdot i}; \quad i = 1, 2, \dots, n. \tag{A5}$$

Since from definition

$$\sum_{j=1}^p \alpha_j m_{ji} = \alpha_{\cdot i} \cdot m_{\cdot i}, \tag{A6}$$

where $\alpha_{\cdot i}$ is the total effect for junction no. i , for each given $\alpha_{\cdot i}$ it is now easy to solve $m_{\cdot i}$ from (A5) by using the method proposed by Danielsson [1986]. If we know α_j we can then solve m_{ji} from

$$m_{ji} = m_{\cdot i} \cdot \frac{x_{ji} + y_{ji}}{x_{\cdot i} + y_{\cdot i} - \alpha_{\cdot i} m_{\cdot i} + \alpha_j m_{\cdot i}}; \quad i = 1, 2, \dots, n, \tag{A7}$$

which are obtained from (A4), (A5), and (A6).

By assigning different α_j and solving the corresponding m_{ji} from (A7) we can test for a solution m_{ji} which must, of course, satisfy (A3).

These p solutions must then satisfy the requirements in (A6) if m_{ji} is to be the ML estimate. If the requirements are not satisfied, new $\alpha_{\cdot i} (i = 1, 1, \dots, n)$ are tested until the optimal solution is obtained.

Thus, it is possible to solve the complicated equation system in (A3) and (A4) by rather simple methods. Numerical solutions are necessary only for calculating all the $m_{\cdot i}$, and in Danielsson [1986] we have presented a method for these calculations.